PHYSICAL REVIEW E, VOLUME 65, 051114

Fraction of uninfected walkers in the one-dimensional Potts model

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(Received 26 November 2001; published 20 May 2002)

The dynamics of the one-dimensional q-state Potts model, in the zero-temperature limit, can be formulated through the motion of random walkers which either annihilate ($A+A\rightarrow\varnothing$) or coalesce ($A+A\rightarrow A$) with a q-dependent probability. We consider all of the walkers in this model to be mutually infectious. Whenever two walkers meet, they experience mutual contamination. Walkers which avoid an encounter with another random walker up to time t remain uninfected. The fraction of uninfected walkers is known to obey a power-law decay $U(t)\sim t^{-\phi(q)}$, with a nontrivial exponent $\phi(q)$ [C. Monthus, Phys. Rev. E **54**, 4844 (1996); S. N. Majumdar and S. J. Cornell, *ibid.* **57**, 3757 (1998)]. We probe the numerical values of $\phi(q)$ to a higher degree of accuracy than previous simulations and relate the exponent $\phi(q)$ to the persistence exponent $\theta(q)$ [B. Derrida, V. Hakim, and V. Pasquier, Phys. Rev. Lett. **75**, 751 (1995)], through the relation $\phi(q)=\gamma(q)\theta(q)$ where γ is an exponent introduced in the preceding paper [S. J. O'Donoghue and A. J. Bray, Phys. Rev. E **65**, 051113 (2002)]. Our study is extended to include the coupled diffusion-limited reaction $A+A\rightarrow B$, $B+B\rightarrow A$ in one dimension with equal initial densities of A and B particles. We find that the density of walkers decays in this model as $\rho(t)\sim t^{-1/2}$. The fraction of sites unvisited by either an A or a B particle is found to obey a power law, $P(t)\sim t^{-\theta}$ with $\theta\simeq 1.33$. We discuss these exponents within the context of the q-state Potts model and present numerical evidence that the fraction of walkers which remain uninfected decays as $U(t)\sim t^{-\phi}$, where $\phi\simeq 1.13$ when infection occurs between like particles only, and $\phi\simeq 1.93$ when we also include cross-species contamination. We find that the relation between ϕ and θ in this model can also be characterized by an exponent γ, where similarly, $\phi=\gamma\theta$.

DOI: 10.1103/PhysRevE.65.051114 PACS number(s): 05.40.−a, 05.50.+q, 82.40.Bj, 05.70.Ln

I. INTRODUCTION

When a system is quenched from a homogeneous high-temperature disordered state into a low-temperature regime, well-defined domains of equilibrium ordered phases form randomly and then grow with time ("coarsen") in a self-similar way, until the domain size becomes comparable to the size of the system [1]. A dynamic scaling hypothesis suggests that, at late times, the evolution of the system is characterized by a single length scale $L(t)$ that represents a typical linear size of the domains. It is well established, at least for systems with a scalar order parameter, that $L(t)\sim t^n$ with $n=1/2$ for *nonconserved* dynamics and $n=1/3$ for *conserved* dynamics [1]. The Ising spin model evolving with Glauber spin-flip dynamics [2] demonstrates behavior characteristic of the former while evolution according to Kawasaki spin dynamics [3] exemplifies the latter. A generalization of the Glauber-Ising model is the q-state Potts model [4–8], which has q distinct, but equivalent, ordered phases. Experimental realizations are known for $q=2$, i.e., the Ising model and for $q=3,4,\infty$ [4]. The $q=\infty$ case describes several cellular structures [9], e.g., polycrystals [10], soap froths [11], magnetic bubbles [12], and foams [13].

Such coarsening systems are among those which have received considerable attention in recent years with regard to their *persistence* properties [13–48]. The persistence probability $P(t)$ is simply the probability that a given stochastic variable $\phi(x,t)$, with zero mean, does not change sign in the time interval $[0,t]$. Theoretical and computational studies of persistence include spin systems in one [15,16] and higher [17–21] dimensions, diffusion fields [22,23], fluctuating interfaces [24], phase-ordering dynamics [25–27], Lotka-Volterra models [28,29] and reaction-diffusion systems [30–38]. Experimental studies include the coarsening dynamics of breath figures [39], foams [13], soap froths [40,41], twisted nematic liquid crystals [42], and one-dimensional gas diffusion [43]. Persistence in nonequilibrium critical phenomena has also been studied in the context of the global order parameter $M(t)$ (e.g., the total magnetization of a ferromagnet), regarded as a stochastic process [44,45]. In many systems of physical interest, the persistence decays algebraically according to $P(t)\sim t^{-\theta}$, where θ is the persistence exponent and is, in general, nontrivial. The nontriviality of θ emerges as a consequence of the coupling of the field $\phi(x,t)$ to its neighbors, since such coupling implies that the stochastic process at a fixed point in space and time is nonMarkovian. For the q-state Potts model at $T=0$, the fraction of spins which have never flipped up to time t (i.e., that fractions that have *persisted* in their original phase), has been observed [15,17–19] to obey a power-law decay $P(t)\sim t^{-\theta(q)}$, where $\theta(q)$ has been obtained exactly, in one dimension, by Derrida *et al.* [16] for arbitrary q

$$\theta(q)=-\frac{1}{8}+\frac{2}{\pi^2}\left[\cos^{-1}\left(\frac{2-q}{\sqrt{2}q}\right)\right]^2. \tag{1}$$

There is a very direct way [49–51] of relating the Glauber dynamics of the one-dimensional (1D) q-state Potts model at zero temperature to reaction-diffusion models [52,53]. Consider uncorrelated initial conditions where each of the q phases is present with equal density $1/q$. During any time interval dt, each spin $S_i(t)$ has a probability $dt/2$ of becoming equal to its right neighbor, $dt/2$ of becoming equal to its left neighbor, and a probability $1-dt$ of retaining its own value. The domain walls therefore perform random walks. Upon contact, two domain walls react instantaneously, either *annihilating* $A+A\rightarrow\varnothing$ with probability $1/(q-1)$ or *coalescing* $A+A\rightarrow A$ with probability $(q-2)/(q-1)$, these numbers being the probabilities that the states on the far sides of the walkers are the same (annihilation) or different (coalescence). This reaction-diffusion model, together with the fact that in the initial condition each bond is occupied by a domain wall with probability $(q-1)/q$, is completely equivalent to the spin problem. The fraction of spins which have never flipped up to time t is then simply the fraction of sites which have never been visited by a random walker. The probabilistic algorithm for implementing the Potts model through the annihilation or coalescence of random walkers allows q to be a real number but is restricted to $q\geqslant 2$. However, an equivalent Ising spin representation of the Potts persistence problem [46] permits a study [37] of $\theta(q)$ in the regime $1<q<2$.

In the present paper, we study the fraction of random walkers which has never encountered another walker, using the reaction-diffusion representation of the 1D Glauber-Potts model at zero temperature. To facilitate our discussion, we consider all the random walkers at time $t=0$ to be uninfected. All the walkers are, however, considered to be infectious, so that for $t>0$ any contact between walkers (assuming they survive the encounter) leads to mutual contamination. Walkers that avoid all contact remain uninfected. Our goal, therefore, is to address the fraction of uninfected walkers $U(t)$ up to time t.

This problem has been addressed in some detail by a number of different authors [26,34,47], although under the guise of a different interpretation. The general approach used involves monitoring the motion of a tagged test particle, released in the system at time $t=0$. If the tagged particle diffuses with diffusion constant D' and the other particles in the system diffuse with diffusion constant D, the tagged particle is viewed as an external impurity diffusing through a homogeneous background. One is then interested in addressing the survival probability of the tagged particle, i.e., the probability that it has not been absorbed by another particle in the system. Of course in our terminology, this corresponds to the probability that the tagged particle remains uninfected. It is well established [34,47] that $U(t)$ decays according to a power law $U(t)\sim t^{-\phi(q,c)}$, where $c=D'/D$ is the ratio of the two diffusion constants in the problem. When $D'=0$, corresponding to a static impurity, the model clearly maps onto the problem of persistent spins and in this case, $\phi(q,0)=\theta(q)$. The exponent $\phi(q,c)$ is therefore considered a generalization of the standard persistence exponent $\theta(q)$, characterizing the survival probability of a mobile particle under the coarsening dynamics of the Potts model. A recent study of the survival probability of a mobile particle, moving according to either deterministic or stochastic rules, in given in [47], where the external fluctuating field takes the form of either the solution to the diffusion equation, the coarsening dynamics of the q-state Potts model, or spatially uncorrelated Brownian signals.

In the limit $q\rightarrow\infty$ the Potts model is equivalent to the $A+A\rightarrow A$ single species coalescence process. The dynamics of this system are particularly simple, since each particle sees only the "cage" formed by its two nearest neighbors. It is immaterial that these neighbors may coalesce with more distant particles. Consequently, the kinetics of each particle involves only itself and its two nearest neighbors and one is therefore able to calculate $\phi(\infty,c)$ exactly [26,34]

$$\phi(\infty,c)=\frac{\pi}{2\cos^{-1}\left[\dfrac{c}{1+c}\right]}. \tag{2}$$

[Note that for $c=0$, this of course reduces to the static persistence exponent, $\phi(\infty,0)=\theta(\infty)=1$.] In the limit of the Ising model $(q=2)$, there is, however, no corresponding exact result for general c. An exact result is only known for $c=0$, i.e., $\phi(2,0)=\theta(2)=3/8$. A mean-field Smoluchowski [26] approach predicts for $q=2$ and general c that, $\phi(2,c)=\sqrt{(1+c)/8}$, in good agreement with numerical simulations [26]. However, when extended to the large q limit, this Smoluchowski approach differs substantially from the exact $q\rightarrow\infty$ result Eq. (2). More recently, however, a perturbation theory developed by Monthus [34] has evaluated the exponent $\phi(q,c)$ to first order in $q-1$ for arbitrary c and at first order in c for arbitrary q. In this paper, we will not study the exponent $\phi(q,c)$ in complete generality since we are primarily interested in the special case where all of the particles are equally mobile $(D'=D)$ and henceforth we therefore write $\phi(q,1)=\phi(q)$.

Within the context of the current paper, where we are interested in the case $c=1$, the random walkers represent the motion of domain walls undergoing zero-temperature coarsening. The fraction of uncollided domain walls (i.e., uninfected walkers) has previously been interpreted as "domain wall persistence" or equally, the probability that two adjacent domains survive. Under this interpretation, Majumdar and Cornell [47] have studied the quantity $U(t)$ numerically for various q. They observed a power-law decay, $U(t)\sim t^{-\phi(q)}$ in agreement with the results of Monthus [34] and obtained a spectrum of values for $\phi(q)$: $1/2\leqslant\phi(q)\leqslant 3/2$, where $2\leqslant q\leqslant\infty$. In this paper, we extend and improve these numerical results by presenting (Sec. II) more accurate data than previously achieved [47] and exploring a larger range of q. We also relate the value of $\phi(q)$ to the static persistent exponent $\theta(q)$ via $\phi(q)=\gamma(q)\theta(q)$, where γ is an exponent that we recently introduced in [57].

We additionally consider (Sec. III) the coupled diffusion-limited reaction $A+A\rightarrow B$, $B+B\rightarrow A$ in one dimension with equal initial densities $\rho_A(0)=\rho_B(0)$. We find that the density of particles decays in this model according to $\rho\sim t^{-1/2}$, independent of the initial walker density, characteristic of "Potts-like" behavior [54]. The fraction of sites unvisited by either an A or a B particle decays as $P(t)\sim t^{-\theta}$ with $\theta\simeq 1.33$. We discuss these exponents within the context of the q-state Potts model and observe that the fraction of walkers which remain uninfected by their *own* species decays according to $U(t)\sim t^{-\phi}$, where $\phi\simeq 1.13$. We also study the fraction of

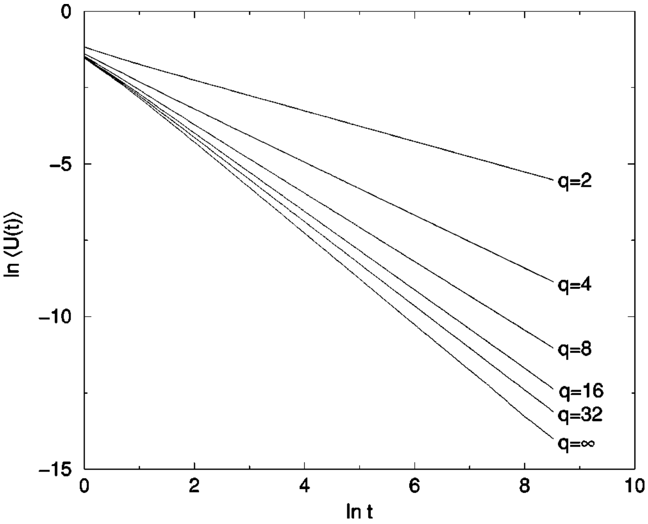

FIG. 1. Log-log plot of the fraction of uninfected walkers in the 1D, $T=0$, Glauber-Potts model as a function of time, for various q.

walkers which remain uninfected by *both* species and find that this too obeys a power law, with $\phi \simeq 1.93$. In analogy to Sec. II, we express the exponent ϕ is terms of the static persistence exponent θ and similarly obtain $\phi = \gamma\theta$. We conclude in Sec. IV, with a summary of the results and a discussion of some open questions.

II. POTTS MODEL

We investigate numerically the fraction of uninfected walkers $U(t)$ in the 1D q-state Potts model, at zero temperature, as a function of q. Our simulations are performed on a 1D lattice of size 10^7 with periodic boundary conditions. At $t=0$ a random walker is placed at each lattice site, so that $\rho(0)=1$. We choose this initial high density of walkers to accelerate the system's evolution into the asymptotic regime. Our model is updated using the direct method [55] i.e., at each computational step, a particle is picked at random and moved with probability $D=1/2$ to a neighboring site, where D is the diffusion constant. If the destination site is occupied, the two particles either annihilate, with probability $1/(q-1)$, or coalesce, to become a single particle, with probability $(q-2)/(q-1)$, in accordance with the reaction-diffusion dynamics of the Potts model detailed in Sec. I. For each move of a particle, time t is increased by $dt=1/N$, where N is the current number of particles in the system. Such sequential dynamics can be chosen without loss of generality as parallel dynamics exhibit similar asymptotic behavior [56]. Our simulations are performed for 5000 time steps and our results are averaged over 100 runs.

TABLE I. Numerical values of the exponent $\phi(q)$ in the 1D, $T=0$, Glauber-Potts model where $U(t) \sim t^{-\phi(q)}$, for various values of q.

q	ϕ
2	0.5006(7)
4	0.867(5)
8	1.121(5)
16	1.283(6)
25	1.352(6)
32	1.380(7)
50	1.419(7)
64	1.435(8)
100	1.461(8)
128	1.464(8)
256	1.481(9)
∞	1.495(9)

Figure 1 clearly shows that the fraction of uninfected walkers decays according to a power-law $U(t) \sim t^{-\phi(q)}$. In Table I, we present our numerical values for $\phi(q)$ in the range $2 \leqslant q \leqslant \infty$. These values were obtained by performing a linear regression on log-log curves such as those presented in Fig. 1. The regression was taken in the range $10 \leqslant t \leqslant 1000$. This range was chosen to avoid initial transients and to avoid statistical fluctuations between different runs which become prominent as t increases, especially for large q. In Fig. 2, we plot ϕ as a function of q, where we choose to represent the data on a linear-log plot merely for clarity of presentation.

The case $q=2$, which corresponds to the Ising model, reduces to the $A+A \rightarrow \emptyset$ reaction-diffusion process, where the density of uninfected walkers is clearly equal to the particle density. It is well known [54] that in the q-state Potts model the particle density obeys a power-law decay $\rho(t) \sim t^{-1/2}$, which is consistent with our value of $\phi(2) = 0.5006(7)$. For the case $q=\infty$, one can obtain the exact value of $\phi(\infty)$ from Eq. (2), i.e., $\phi(\infty,1)=3/2$. This is again consistent with our numerical result, $\phi(\infty)=1.495(9)$. These limiting values, along with our numerics, suggest that the fraction of uninfected walkers decays according to $U(t) \sim t^{-\phi(q)}$ where $1/2 \leqslant \phi(q) \leqslant 3/2$, for $2 \leqslant q \leqslant \infty$, consistent with the results of Monthus [34] and Majumdar and Cornell [47]. Although the values of $\phi(q)=1/2$ and $\phi(\infty)=3/2$ at the limit of the q spectrum are understood, it was pointed out

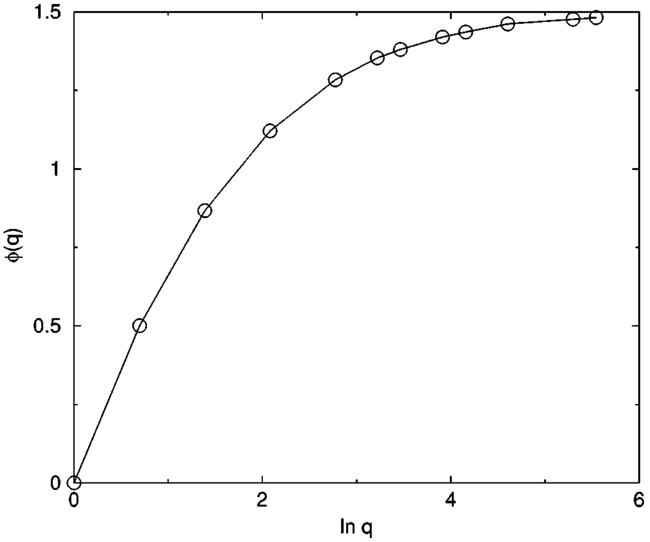

FIG. 2. Linear-log plot of ϕ as a function of q for the 1D, $T=0$, Glauber-Potts model.

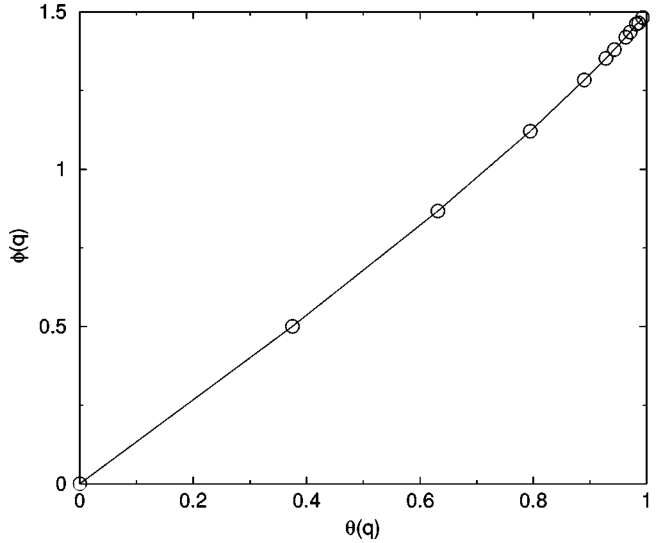

FIG. 3. The exponent ϕ plotted against the exponent θ, for various q for the 1D, T=0, Glauber-Potts Model.

by Monthus [34] that the formalism that led to the exact determination of the static persistence exponent $\theta(q)$ [16] cannot be easily applied to compute the exact value of $\phi(q,c)$ for general c, including the limit $c=1$. The exact calculation of $\phi(q)$ therefore remains an open problem.

Recently, we have explored the connection between uninfected walkers and unvisited sites in one dimension within the context of a system of noninteracting randomly diffusing particles $A\emptyset \leftrightarrow \emptyset A$, and the $A+B\rightarrow\emptyset$ diffusion-limited reaction [57]. We found that in both of these models

$$U(t)\simeq[P(t)]^{\gamma}, \quad \text{with } \gamma\simeq 1.39. \tag{3}$$

While this relationship is obeyed reasonably accurately over the range of times accessible to our simulations, there is evidence, for both processes, that γ approaches a smaller value at late times [57]. Clearly, we can write an analogous relationship for the q-state Potts model, where $U(t)\sim t^{-\phi(q)}$, $P(t)\sim t^{-\theta(q)}$, and γ is now some function of q, i.e., $t^{-\phi(q)}=t^{-\gamma(q)\theta(q)}$, so that

$$\phi(q)=\gamma(q)\theta(q). \tag{4}$$

We plot $\phi(q)$ against $\theta(q)$ in Fig. 3, where our values for $\phi(q)$ have been taken from Table I and the corresponding values of $\theta(q)$ have been calculated exactly from Eq. (1). The point at the origin corresponds to $q=1$, where ϕ and θ both vanish. Figure 3 indicates that $\phi(q)$ increases monotonically as a function of $\theta(q)$. Therefore, given that we know the exact values of $\theta(q)$ and $\phi(q)$ for $q=2,\infty$, and given the convex nature of the plot in Fig. 3, we can use Eq. (4) to determine precise bounds on $\gamma(q)$ in this range

$$4/3\leqslant\gamma(q)\leqslant 3/2, \quad q\in[2,\infty]. \tag{5}$$

Thus, while $\phi(q)$ and $\theta(q)$ are separately quite strongly dependent on q, their ratio $\gamma(q)$ depends only weakly on q.

III. EXCHANGE MODEL

The $A+A\rightarrow B$, $B+B\rightarrow A$ diffusion-limited reaction offers a curious combination of birth and death processes. For brevity, we dub this model the "exchange model." The particles in the exchange model diffuse randomly. Whenever a particle encounters a walker belonging to its own species, the two particles annihilate instantaneously, to create a particle belonging to the *other* species at the point of annihilation. For the purposes of this paper, we allow lattice sites to contain multiple particles, enabling two particles of differing species to cohabit a single lattice site. This feature of our model ensures that, in one dimension, the lattice structure $\cdots ABABABABABAB\cdots$ continues to evolve. Confining our system to a maximum of one particle per site would otherwise mean that the preceding configuration would be stable for all time t. We focus our attention on the 1D exchange model with equal initial densities of A and B particles.

Our numerical simulations are performed on a one-dimensional lattice with periodic boundary conditions. At $t=0$ equal numbers of A and B particles are randomly distributed on the lattice with a maximum of one particle per site. Our system then evolves according to the dynamics described above. The system is updated in the same manner as for the q-state Potts model described in Sec. II.

A. Particle density

The first question we address is the particle decay. The unimolecular reactions defining the exchange model are characteristic of the q-state Potts model and therefore we can expect the particle density to exhibit similar asymptotic behavior. Indeed, for every pair of reactions between two A and two B particles, a single A and a single B are produced, which is the same result as for two $q=\infty$ Potts models $A+A\rightarrow A$, $B+B\rightarrow B$ operating on a single lattice. We therefore expect the particle density to decay in the exchange model according to $\rho_A(t)\sim Ct^{-\alpha}$, with $\alpha=1/2$, independent of the initial particle density [54]. This is indeed confirmed by our simulations. In Fig. 4, we present our results for the particle density for three different initial values of $\rho_A(0)$ [recall $\rho_B(0)=\rho_A(0)$]. The simulations were performed on a lattice of size 10^6 for 10^5 time steps and the results averaged over 100 runs.

In Table II, we present our numerical values for the particle density decay exponent α and amplitude C, where $\rho_A(t)\sim Ct^{-\alpha}$. These values were obtained by performing a linear regression on the plots in Fig. 4 in the asymptotic regime, $10^4\leqslant t\leqslant 10^5$.

For the 1D q-state Potts model, the amplitude C is q dependent [54]

$$C(q)=\frac{q-1}{q}\frac{1}{\sqrt{2\pi D}}. \tag{6}$$

Is there perhaps a value of q which corresponds to the exchange model? A value of $q=4$ in Eq. (6), with $D=1/2$ returns a value $C(4)=0.423$, in good agreement with the

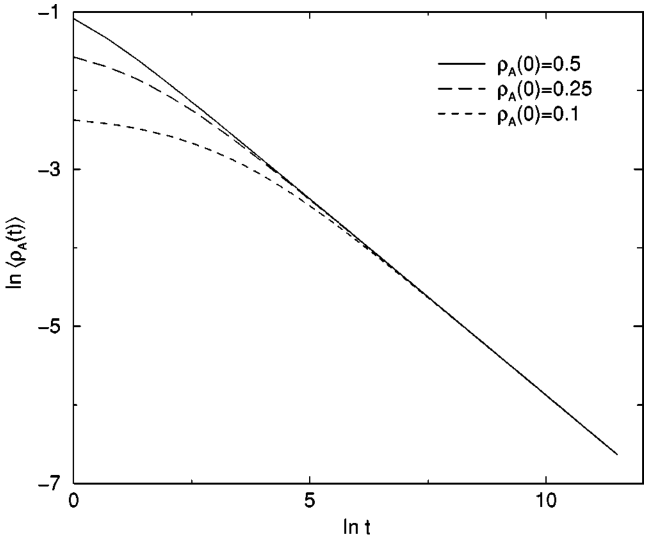

FIG. 4. Log-log plot of the particle density as a function of time in the 1D exchange model.

values of C recorded in Table II. Notice that this value lies exactly midway between the minimum value $C(2)=0.282$ and the maximum value, $C(\infty)=0.564$, set by the q-state Potts model. Our numerical results therefore suggest that in terms of the particle density, the two reactions which define the exchange model ($A+A\rightarrow B, B+B\rightarrow A$) are equivalent, in some loose sense, to two independent $q=4$ Potts models, though we would not wish to stretch this comparison too far.

B. Fraction of unvisited sites

We now turn our attention to the persistence properties of the exchange model. Here, we are interested in the fraction of sites $P(t)$ that have never been visited by either an A or a B particle up to time t. An obvious starting point is to see if $P(t)$ obeys a power-law decay with an exponent equal to that returned in the case of two $q=4$ Potts Models, i.e., $P(t) \sim t^{-2\times\theta(q=4)}$ where, using Eq. (1), $2\times\theta(q=4)=1.263$. Our numerical simulations do indeed indicate a power-law decay $P(t)\sim t^{-\theta_{ex}}$. However, the decay exponent $\theta_{ex}\simeq 1.33$, in disagreement with what one might have naively expected from a study of the particle decay.

Our simulations are performed on a lattice of size 10^6 for 5000 time steps and the results averaged over 200 runs. In Fig. 5, we present a log-log plot of $P(t)$. In Table III, we show the results of a linear regression performed on the curves in Fig. 5, to obtain values of θ_{ex}. The regression was performed in the regime $500\leqslant t\leqslant 5000$, to avoid initial transients.

TABLE II. Numerical values for the particle density decay exponent α and the amplitude C in the 1D exchange model, where $\rho_A(t)\sim Ct^{-\alpha}$, for different initial densities $\rho_A(0)$.

$\rho_A(0)$	α	C
0.5	0.5007(9)	0.419(5)
0.25	0.5007(9)	0.419(5)
0.1	0.5000(5)	0.416(9)

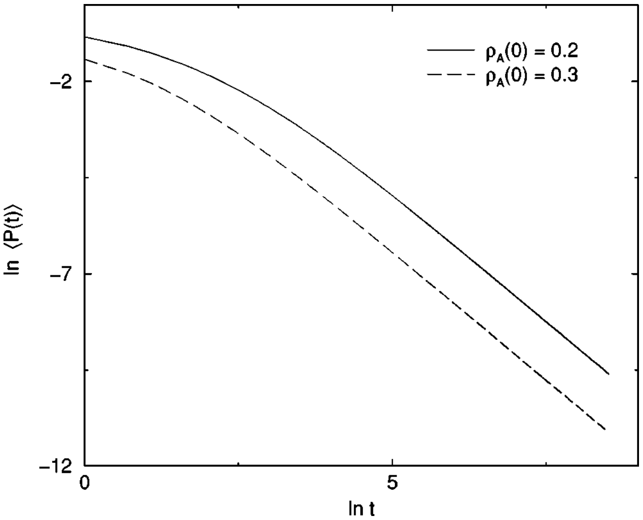

FIG. 5. Log-log plot of the fraction of unvisited sites as a function of time in the 1D exchange model.

C. Fraction of uninfected walkers

Finally, we consider the fraction of uninfected walkers in the exchange model. We focus, without loss of generality, on just the fraction of uninfected A particles. In this model, we are only interested in *original* A particles. Those A particles which have been created through the reaction $B+B\rightarrow A$ after $t=0$, are also therefore considered to be infected. There are two distinct cases to consider: (i) the fraction of A particles which remain uninfected by *only* their own species and (ii) the fraction of A particles which have avoided infection by *both* types of species. Our approach is wholly numerical. We address the former case first.

Figure 6 indicates that the fraction of uninfected A walkers obeys a power-law decay according to $U(t)\sim t^{-\phi_A}$, where the subscript A on the exponent denotes infection by only A particles [case (i)]. The results for $\rho_A(0)=1$ were generated from 500 runs on a lattice of size 5×10^5 over 5000 time steps. The results for $\rho_A(0)=0.5$ were generated using a lattice of size 10^6, run for 10^4 time steps and also averaged over 500 runs. In Table IV, we present our numerical values of ϕ_A for two different initial starting densities. The values of the exponent were obtained by performing a linear regression on the curves in Fig. 6 in the regime $100 \leqslant t\leqslant 5000$.

We now address case (ii), the fraction of A particles which have avoided infection by both A and B walkers. Our simu-

TABLE III. Numerical values for the persistence decay exponent θ_{ex} in the 1D exchange model, where $P(t)\sim t^{-\theta_{ex}}$, for initial densities $\rho_A(0)$.

$\rho_A(0)$	θ_{ex}
0.2	1.330(7)
0.3	1.333(5)

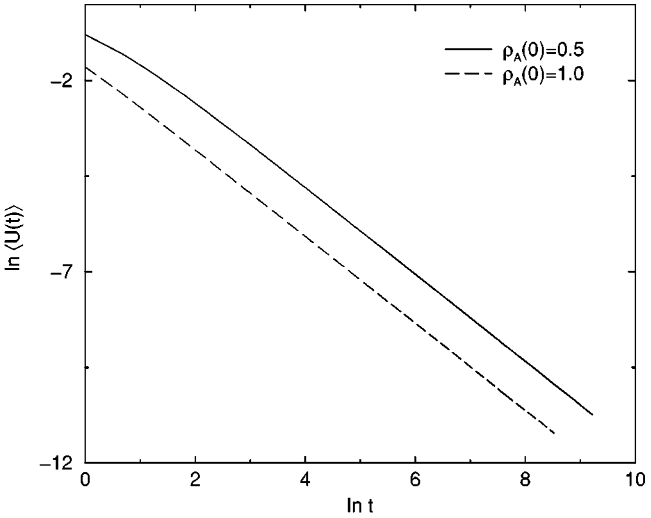

FIG. 6. Log-log plot of the fraction of *A* walkers uninfected by *A* particles in the 1D exchange model as a function of time.

lations are performed on a lattice of size 10^6 for 2000 time steps and averaged over 500 runs. In line with our previous data, we present our results for $U(t)$ on a log-log plot in Fig. 7.

Unfortunately, we do not obtain straight line graphs. However, an inspection of individual runs of the data (Fig. 8) shows that, in this model, statistical fluctuations between different runs become prominent very quickly, whereas earlier times are governed by transient effects. The dominance of these two regimes renders any determination of an exponent particularly difficult. Using a lower initial density, with the intention of delaying the onset of noisy data only extends the transient regime, whereas using a higher initial density to minimize transient effects accelerates the onset of large statistical fluctuations. In view of the power-law behavior we identified in case (i) and the general trend of our data, we suggest that, with sufficiently good statistics, the fraction of *A* walkers uninfected by either *A* or *B* particles would also exhibit a power-law decay $U(t)\sim t^{-\phi_{AB}}$, where the subscript on the exponent denotes infection is permitted by both species. There is, however, a cleaner method of extracting the exponent ϕ_{AB}. We suggest, in analogy with the other models we have studied, that $U(t)\simeq[P(t)]^{\gamma_{ex}}$ so that

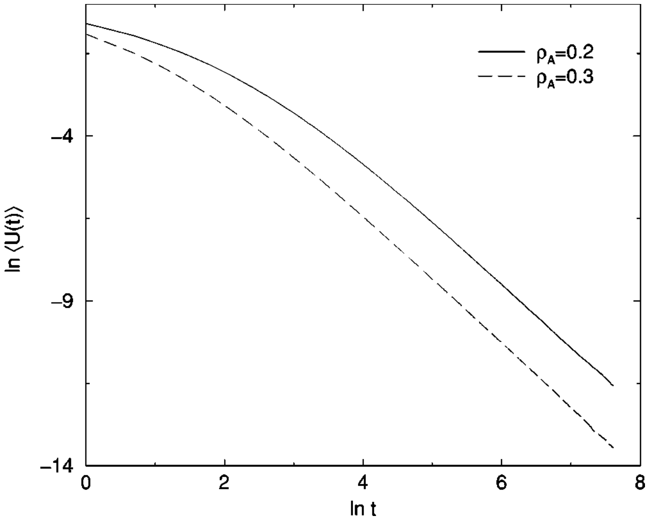

FIG. 7. Log-log plot of the fraction of *A* walkers uninfected by both *A* and *B* particles in the 1D exchange model as a function of time.

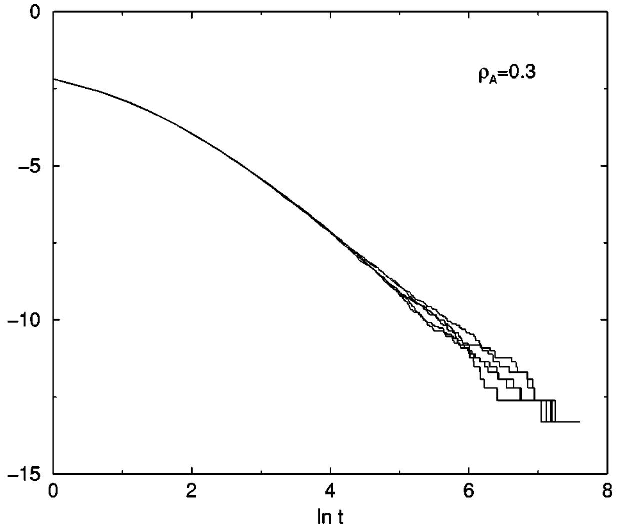

FIG. 8. Log-log plot of the fraction of *A* walkers uninfected by both *A* and *B* particles in the 1D exchange model as a function of time, for five individual simulation runs.

$$\phi_{AB}=\gamma_{ex}\theta_{ex}. \tag{7}$$

If we can establish the validity of the relation $U(t)\simeq[P(t)]^{\gamma_{ex}}$, and determine the value of γ_{ex}, we can then read off the exponent ϕ_{AB} from Eq. (7). Our initial condition specifies a maximum of one particle per site, so $U(0)=1$. While, in the continuum description of the model, $P(0)=1$ also, on a lattice this becomes $P(0)=1-\rho_0$. Therefore, to place these two quantities on an equal footing in our numerical study, we define, $p(t)=P(t)/P(0)$, so that both $U(0)$ and $p(0)$ are equal to unity. Note, of course, that in the continuum limit, $\rho_0\to 0$, we recover $p(t)\to P(t)$. We present in Fig. 9 a log-log plot of $U(t)$ vs $p(t)$.

The plot in Fig. 9 is quite linear, but does not go through the origin, suggesting that in fact $U(t)\simeq Kp(t)^{\gamma_{ex}}$ with $K\neq 1$. We note that in the numerical analysis of the relationship between noninfectedness and persistence in the $A\varnothing\leftrightarrow\varnothing A$ and $A+B\to\varnothing$ models [57] the equivalent prefactor K was close enough to unity ($K\simeq 0.99$) to tempt the speculation that it might become unity in the continuum (low-density) limit, though we stress that in the latter models there was good evidence that some of the simulations had

TABLE IV. Numerical values for the exponent ϕ_A in the 1D exchange model, where $U(t)\sim t^{-\phi_A}$, for initial densities $\rho_A(0)$.

$\rho_A(0)$	ϕ_A
1.0	1.133(5)
0.5	1.138(8)

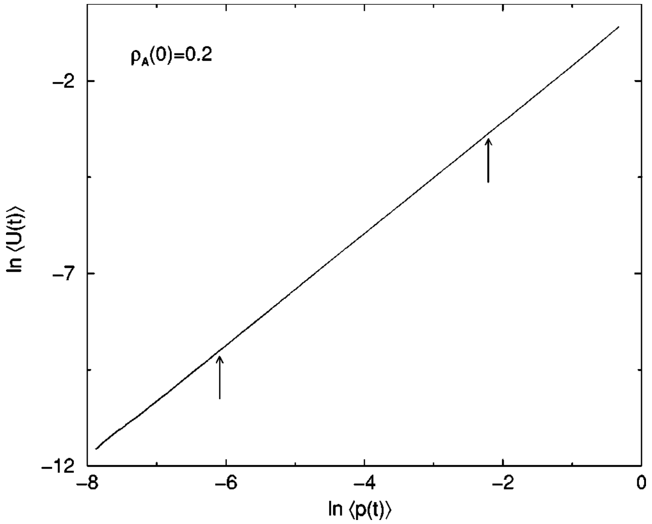

FIG. 9. Log-log plot of the fraction of A walkers uninfected by both A and B particles, against the fraction of unvisited sites, in the 1D exchange model.

not reached the asymptotic regime. In order to evaluate γ_{ex} and K for the present model, we perform a linear regression on the data in Fig. 9 in the region indicated by the arrows, thereby avoiding initial transients associated with lattice effects, and the onset of statistical fluctuations between different runs of the data. We summarize our results for γ_{ex} and K in Table V.

Note that the value of K for $\rho_A(0)=0.2$ is larger than that for $\rho_A(0)=0.3$, suggesting the possibility that $K\rightarrow 1$ as $\rho_A(0)\rightarrow 0$, consistent with our results for other models [57], though we have no concrete arguments to support this idea. To test our results, we attempt to collapse the data for the uninfected fraction and the persistence on to a single curve by plotting $\ln U(t)$ and $\gamma_{ex}\ln p(t)+\ln K$ as functions of time t, in Fig. 10.

The data collapse is excellent over both transient and asymptotic regimes, suggesting that the relation $U(t)\simeq K[p(t)]^{\gamma_{ex}}$ holds to a high degree of accuracy for all times t. We attribute any deviations between the two curves at late times to the onset of statistical fluctuations between different runs of the data. Clearly we can improve the collapse at either early or late times by performing the linear regression in Fig. 9 on the appropriate part of the curve. We find that over the whole regime presented in Fig. 10, $\gamma_{ex}\simeq 1.45(1)$ works well. For brevity and clarity of presentation, we have presented our results for $\rho_A(0)=0.2$ only, although we have achieved equally good results for $\rho_A(0)=0.3$.

The values $\gamma_{ex}\simeq 1.45$ and $\theta_{ex}\simeq 1.33$ yield, using Eq. (7), $\phi_{AB}\simeq 1.45\times 1.33\simeq 1.93$. We therefore argue that, when infection is caused by both species, $U(t)\sim t^{-\phi_{AB}}$ with $\phi_{AB}\simeq 1.93$.

TABLE V. Numerical values of γ_{ex} and K in the 1D exchange model, where $U(t)\simeq K[p(t)]^{\gamma_{ex}}$, for two initial densities $\rho_A(0)$.

$\rho_A(0)$	γ_{ex}	K
0.2	1.453(7)	0.863(9)
0.3	1.456(7)	0.754(9)

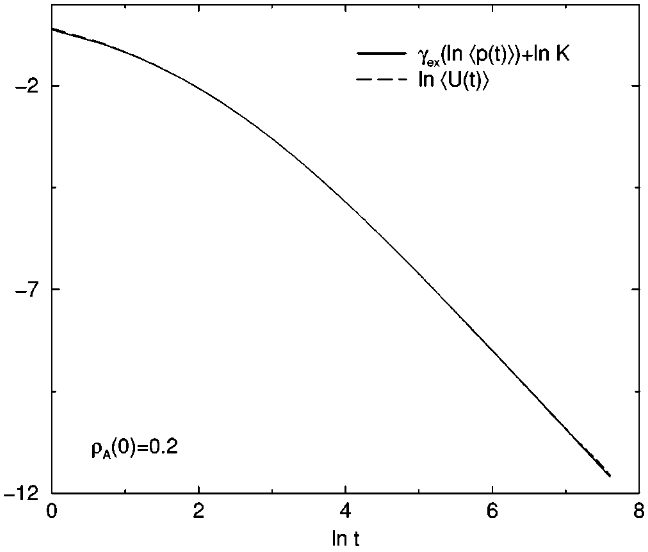

FIG. 10. $\ln U(t)$ and $\gamma_{ex}\ln p(t)+\ln K$ plotted as a function of time, to collapse the data for the uninfected walkers and unvisited sites onto a single curve. We present our results for $\rho_A(0)=0.2$ and use $\gamma_{ex}=1.45$ and $K=0.86$.

IV. DISCUSSION AND SUMMARY

In this paper, we have investigated the fraction of uninfected walkers $U(t)$ in the 1D q-state Potts model evolving at zero temperature. Our numerical results are consistent with previous work [26,34,47], suggesting that $U(t)\sim t^{-\phi(q)}$ with a nontrivial exponent $\phi(q)$, where $1/2\leqslant\phi(q)\leqslant 3/2$ for $2\leqslant q\leqslant\infty$. Although the values of $\phi(q)$ for $q=2$ and $q=\infty$ are understood, the exact calculation of $\phi(q)$ for all q, appears to be a hard problem. In analogy with the other models we have studied [57], we have reduced the familiar study of persistent sites $P(t)$, to a limiting case in the study of uninfected walkers and noted that $U(t)\sim[P(t)]^{\gamma(q)}$, with $4/3\leqslant\gamma\leqslant 3/2$ for $2\leqslant q\leqslant\infty$. We discussed in Sec. I that, with regard to the persistence properties of the Potts model, it is possible to probe the region $1\leqslant q\leqslant 2$. Although we have included a point for the state $q=1$, where $\theta=\phi=0$, in Figs. 2 and 3, we leave a study of the fraction of uninfected walkers in this latter regime as an open challenge.

We have also studied the 1D exchange model, defined by the reactions $A+A\rightarrow B$ and $B+B\rightarrow A$. Our numerical simu-

TABLE VI. Numerical results for the exponent γ in various 1D models for which $U(t)\simeq[P(t)]^{\gamma}$ (results for the first two models from Ref. [57]).

Model	γ
$A\emptyset\leftrightarrow\emptyset A$	1.39
$A+B\rightarrow\emptyset$	1.39
q-state Potts Model	[4/3,3/2], $q\in[2,\infty]$
exchange Model	1.45

lations indicate similarities between this diffusion-limited system and the Potts class of models, in that the particle density decays according to $\rho(t) \sim t^{-1/2}$, independent of the initial particle density. The amplitude of the particle decay lies approximately halfway between the minimum and maximum values set by the Potts model suggesting that, within the context of the particle decay, the exchange model roughly mimics the behavior of two independent $q=4$ Potts models. We have also shown that the fraction of sites unvisited by either an A or a B particle decays according to $P(t) \sim t^{-\theta_{ex}}$ where $\theta_{ex} \simeq 1.33$. Our study of the fraction of uninfected walkers in the exchange model focused on two cases. In the first instance, we addressed the fraction of A walkers infected by only their own species and established that, in this case, $U(t) \sim t^{-\phi_A}$ where $\phi_A \simeq 1.13$. When cross-species contamination was also included in our simulation, we found $U(t) \sim t^{-\phi_{AB}}$ with $\phi_{AB} \simeq 1.93$. We presented numerical evidence that $U(t) \simeq K[p(t)]^{\gamma_{ex}}$ with $\gamma_{ex} \simeq 1.45$. An obvious goal for the future is to establish a theoretical framework for the exponents that occur in the exchange model and, in particular, for the observed simple relation between $U(t)$ and $P(t)$. Across the range of models we have studied, in both this and our previous paper [57], the fraction of uninfected walkers $U(t)$ and the fraction of unvisited sites $P(t)$ are seemingly related by $U(t) \sim [P(t)]^{\gamma}$. This remarkable relationship, which seems to hold reasonably accurately across the entire time regime accessible to simulation, merits further attention. In Table VI, we summarize the values of γ for the various models we have studied (note, however, that there is some evidence [57] that, for the first two models, the asymptotic value of γ is smaller 1.39).

Finally, we note that all values of γ obtained so far lie in the range $4/3 \leqslant q \leqslant 3/2$ spanned by the q-state Potts model with $2 \leqslant q \leqslant \infty$. Up to now, our analysis of γ has been purely numerical, and naturally a more fundamental understanding of this exponent is required.

ACKNOWLEDGMENTS

We thank Satya Majumdar for drawing our attention to some related work. This work was supported by EPSRC (UK).

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